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Robustness analysis of geodetic horizontal networks

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Abstract. Traditional reliability analysis has been augmented with geometrical strength analysis using strain techniques, resulting in the conception of an extension to reliability theory called robustness analysis. To reflect contemporary statistical terminology, robustness is taken to mean insensitivity to gross errors or blunders in the data. Robustness analysis is a natural merger of reliability and strain and is defined as the ability to resist deformations induced by the smallest detectable blunders as determined from internal reliability analysis. The geometrical strength analysis technique is used in order to provide a more complete and detailed description of the potential network deformation in terms of three independent measures representing robustness in scale, orientation, and configuration. These measures are also invariant with respect to "datum" shifts and orientation, and practically invariant to changes in scale. Initial experiences with robustness analysis have shown that it is a very powerful technique capable of providing a detailed point-by-point assessment of the strength of a network.

Key words: Statistical Testing – Type II Errors – Robustness – Strength Analysis – Reliability Theory – Strain Analysis

1 Introduction

In most countries, horizontal geodetic networks are tested only in the statistical sense. Statistical testing consists of: testing for gross errors in observations; testing of a posteriori value of the variance factor σ_0^2 (often incorrectly called "a posteriori value of standard deviation of unit weight"); testing of absolute and relative confidence ellipses, individually and collectively; and testing of a posteriori estimates of residuals (Vaníček and Krakiwsky 1986). All these tests are based on the same null hypothesis H_o which postulates the vector of misclosures $\mathbf{w} = \mathbf{l} - \mathbf{f}(\mathbf{x}^{(0)})$ to have a normal probability distribution with an expected mean value $\mathbf{A}\ \delta\mathbf{x}$ (where $\mathbf{A}$ is the matrix of partial derivatives of observations $\mathbf{l}$ with respect to $\mathbf{x}$ and $\delta\mathbf{x}$ is the vector of corrections to the initial value $\mathbf{x}^{(0)}$ of $\mathbf{x}$) and expected covariance matrix $\mathbf{C}_\mathrm{l}$

$$H_o\text{: } \mathbf{w} \text{ is distributed as } n\ (\xi; \mathbf{A}\ \delta\mathbf{x}, \mathbf{C}_\mathbf{l}) \qquad (1)$$

Also common to all the tests is the selection of an a priori value α_o, called the significance level of the test, which is the level of probability that the hypothesis would be rejected when it is correct, i.e. the probability of type I error. This individual or so-called in-context probability level must be compatible with the simultaneous probability level used for the global variance factor test on the residuals. For an α simultaneous probability level and n observations, the individual significant level α_o is $\alpha_o \approx \alpha/n$ (this is a crude approximation valid only for small n; Vaníček and Krakiwsky 1986).

The additional question we should also pose is: what would happen if one or more observations, $\mathbf{l}$, are burdened with a gross error $\Delta\mathbf{l}$ so that the null hypothesis of Eq. (1) is not satisfied. Evidently, if $\Delta\mathbf{l}$ is such that it gets detected by the statistical test on estimated residuals on α_o significance level, the erroneous observation can be corrected (in practice more likely deleted), the network re-adjusted, and we obtain correct results. The problem occurs when $\Delta\mathbf{l}$ is not detected by the test. This may happen because of one of two reasons: (i) the observation is not sufficiently checked by other independent observations (e.g. in a one-sided traverse); or (ii) the test does not recognize the gross error, i.e. it commits a type II error.

It was a Dutch geodesist, Baarda, who first investigated the problem of type II errors in geodetic networks. The results of his investigations are summarized in his publications on the theory of reliability; see, for example, Baarda (1968). The central idea of Baarda's theory of reliability is the formulation of an alternative hypothesis H_A,

$$H_A\colon \mathbf{w} \text{ is distributed as } n\ (\xi; \mathbf{A}\ \delta\mathbf{x} + \Delta\mathbf{l}, \mathbf{C_l}) \tag{2}$$

which postulates the existence of gross errors $\Delta\mathbf{l}$. (In Baarda's original formulation the alternative hypothesis looks a little different from the form used here.) By means of reliability theory we can estimate the maximum values of gross error $\Delta l_{\max,i}$ in observations, which would not be detected by a statistical test on α_o significance level, or, conversely, we can estimate the probability of a given type II error occurring. These estimates are governed by the relation

$$\Delta l_{\max,i} = \lambda_o(\alpha_o, \beta_o)\frac{\sigma_l}{\sqrt{r}} \tag{3}$$

where λ_o is the value of the shift (non-centrality parameter) of the postulated distribution in the alternative hypothesis of Eq. (2) as a function of selected probabilities α_o, β_o; $\sigma_{l,i}$ is the a priori value of standard deviation of the ith observation l_i (considered known), and $r_i \in \langle 0, 1\rangle$ is Baarda's redundancy number, which expresses the degree of influence on the estimated (adjusted) positions of the ith observation (Baarda 1968). Baarda refers to the quantities $\Delta l_{\max,i}$ as a measure of internal reliability. Figure 1 illustrates the relation between α_o, β_o, and λ_o.

For completeness, we have re-stated briefly the classical approach to reliability because it is one of the building blocks of the "robustness analysis". As soon as we know how to estimate the maximum values of errors that can be expected to pass the statistical tests, it makes sense to ask a follow-up question: how much can these undetected errors influence the estimated positions? When the influence is small, we talk of a "robust" network; if the influence is large, we talk of a weak network that lacks robustness. Measuring the degree of such "robustness" is the aim of what we call "robustness analysis". We have selected the term "robustness" to reflect its modern usage in statistics, i.e. those statistics that are insensitive to outliers are called robust. Here, our robustness analysis provides a direct measure of sensitivity to outliers. It seems to us that this term is more intuitively descriptive than any of the possible alternatives, such as "external reliability" or "network strength".

We have studied the robustness of networks for some time and published some of our findings in a paper (Vaníček 1991), in a series of technical reports (Vaníček et al. 1991, 1996; Krakiwsky et al. 1993), and in conference presentations (Craymer et al. 1993a, b, 1995; Szabo et al. 1993). Here, we endeavour to summarize our findings as well as to provide an explicit proof for the robustness datum independence.

2 Description of network deformation

In order to be able to measure the degree of robustness of a network, we have to be able to measure its degree of deformation. The most simple way of describing a deformation is by means of displacements of individual points of the network. On the basis of normal equations (Vaníček and Krakiwsky 1986), we can write the following expression for the adjusted coordinates as linear combinations of collected observations:

$$\hat{\mathbf{x}} = \mathbf{x}^{(0)} + \delta\hat{\mathbf{x}} = \left(\mathbf{A}^T\mathbf{C}_l^{-1}\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{C}_l^{-1}\left(\mathbf{l} - \mathbf{F}(\mathbf{x}^{(0)})\right) \tag{4}$$

and easily derive also the estimates for displacements $\Delta\hat{\mathbf{x}}$ caused by arbitrary changes $\Delta\mathbf{l}$ in observations

$$\Delta\hat{\mathbf{x}} = (\mathbf{A}^T\mathbf{C}_l^{-1}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{C}_l^{-1}\Delta\mathbf{l} \tag{5}$$

This was Baarda's choice – actually he chose to use a norm of the displacements $\Delta\hat{\mathbf{x}}$ and misleadingly referred to it as a measure of external reliability (it is really only a measure of internal reliability but in parameter space rather than observation space).

The problem with displacements is that their estimates are datum dependent. That is, these estimates depend not only on the geometry of the network, and the accuracy of the observations, but also on the selection of constraints for the adjustment (fixed point, weighted point, fixed or weighted azimuth, minimum constraint criterion, generalized inversion of the matrix of normal equations, etc.), which have nothing to do with the network deformation. If we want to use the described deformation for the quantification of network robustness then the deformation description must reflect only network geometry, and the type and accuracy of the observations. The deformation description must therefore be independent of adjustment constraints (datum)! Strain, which describes a differential deformation, is one such description. Let us denote a displacement of a point P_i by

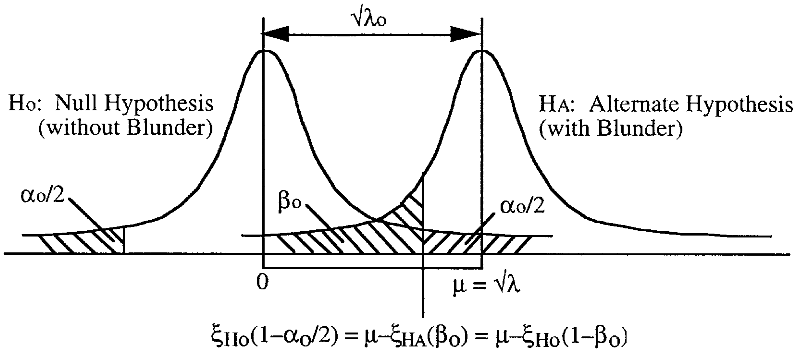


Fig. 1. Relationship between α_o, β_o and λ_o. Note the relationship between λ_o and the abscissa values ξ for H_o and H_A: i.e. $\lambda_o = \xi\ (1\alpha_o/2) + \xi\ (1\beta_o)$

$$\Delta \mathbf{x}_i = \begin{bmatrix} \Delta x_i \\ \Delta y_i \end{bmatrix} = \begin{bmatrix} u_i \\ v_i \end{bmatrix} \tag{6}$$

then the tensor gradient with respect to position is

$$\mathbf{E}_i = \mathrm{grad}(\Delta \mathbf{x}_i) = \begin{bmatrix} \partial u_i/\partial x & \partial u_i/\partial y \\ \partial v_i/\partial x & \partial v_i/\partial y \end{bmatrix} \tag{7}$$

where $\Delta \mathbf{x}_i$ is the displacement vector of P_i. Matrix $\mathbf{E}$ is the so-called deformation or strain matrix (at point P_i) and is independent of the choice of the "fixed" (or weighted) point for the adjustment (Vaníček and Krakiwsky 1986). It is of interest to note that the following relation holds:

$$\Delta \mathbf{x}_i = \mathbf{E}_i \mathbf{x}_i + \mathbf{c}_o \tag{8}$$

where $\mathbf{x}_i$ is the position vector of point P_i and $\mathbf{c}_o$ is an arbitrary shift vector, constant for the network. The deformation matrix is usually decomposed into its symmetric and anti-symmetric parts

$$\mathbf{E} = \mathbf{S} + \mathbf{A} \tag{9}$$

where

$$\mathbf{S} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{bmatrix} \tag{10}$$

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \tag{11}$$

The matrix $\mathbf{S}$ describes symmetrical differential deformation at a point which is often used in continuum mechanics (cf. Means 1976). The symbol ω in the $\mathbf{A}$ matrix (not to be confused with the already introduced design matrix) describes a differential rotation at the point of interest. This differential rotation can be further decomposed into the (block) rotation ω_o common to the whole network

$$\omega_o \approx \text{mean } (\omega_i) \tag{12}$$

and the proper differential rotation $\delta\omega$ at each point.

$$\delta\omega_i \approx \omega_i - \text{mean } (\omega_i) \tag{13}$$

Equations (7)–(13) are valid for any point of a continuum, and $\mathbf{S}$, $\mathbf{A}$, ω, and $\delta\omega$ can be regarded as compact functions of positions. For our applications, however, it makes sense to define them only for the points of the network, i.e. as point functions.

We do not claim to be the first to use strain for geodetic applications. Probably the first application of strain in geodesy is due to Tsuboi (1932), who used strain to portray horizontal crustal deformation in the Tango area in Japan. For more information on strain analysis, see Vaníček and Krakiwsky (1986, pp 649–655) and references therein. For more recent references see, for example, Grafarend (1985), Dermanis and Grafarend (1993), and Welsch (1997).

3 Datum independence of the deformation matrix

The effect of the adjustment constraints, or datum definition, on the deformation matrix is an important issue in the strength analysis of networks. The origin is usually defined by specifying fixed or heavily weighted coordinates for one or several points in the network. The orientation is often defined using weighted observations such as azimuths or observed position differences between points. The scale datum is generally defined using weighted distances or, again, position difference observations. Two or more weighted position observations are also used to define datum orientation and scale.

Ideally the strength of a network should not depend on the choice of a datum. It will be shown here that a scale change has only a second-order and thus negligible effect on the deformation matrix, while translations of the datum origin and rotations of the coordinate system have no effect at all. We must stress, however, that we are talking about a datum specified by minimal constraints. If more than minimal constraints are used, we are faced with a very different problem, that of an over-constrained solution. An over-constrained solution will indeed show a deformation different from that of a minimally constrained solution. Discussion of the additional deformation due to introduced constraints over and above the minimal ones, is outside the scope of this paper.

Let us discuss the effect of datum origin first. Differences in the datum origin between different strain (or strength) solutions completely cancel in the determination of the blunder-caused displacements. That is, the displacements for any two solutions are identical even though they may be based on datums with different origins. Consider the estimated position vectors $\mathbf{x}_1$ from observations without any blunder and the position vectors $\mathbf{x}_2$ after the network has been perturbed by one or several blunders. The displacements $\Delta\mathbf{x}$ caused by the blunders are then

$$\Delta \mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1 \tag{14}$$

Consider now a second adjustment using a different datum origin, which is offset from that for the first solution by a translation $\mathbf{t}$. For this solution, the original coordinates $\mathbf{x}_1^*$ and those $\mathbf{x}_2^*$ after the perturbation by the same blunders, can be expressed in terms of the coordinates for the first solution as

$$\mathbf{x}_1^* = \mathbf{x}_1 + \mathbf{t} \tag{15a}$$

$$\mathbf{x}_2^* = \mathbf{x}_2 + \mathbf{t} \tag{15b}$$

The displacement field $\Delta\mathbf{x}^*$ (caused by the same blunders) for this second solution is then

$$\Delta \mathbf{x}^* = \mathbf{x}_2^* - \mathbf{x}_1^* = \mathbf{x}_2 - \mathbf{x}_1 = \Delta \mathbf{x} \tag{16}$$

which is identical to the first solution. Any translation of the datum origin therefore cancels in the displacements. Since the displacements are identical in both solutions, the deformation matrices will also be identical and thus invariant to datum translations.

Next, let us look at the effect of the scale constraint. The adjustment before and after a change in scale of Δs results in the following coordinates:

$$\mathbf{x}_1^* = (1 + \Delta s)\mathbf{x}_1 \tag{17a}$$

$$\mathbf{x}_2^* = (1 + \Delta s)\mathbf{x}_2 \tag{17b}$$

The displacement field $\Delta\mathbf{x}^*$ for this second solution is then

$$\Delta\mathbf{x}^* = \mathbf{x}_2^* - \mathbf{x}_1^* = (1 + \Delta s)\Delta\mathbf{x} \tag{18}$$

and the corresponding strain matrix $\mathbf{E}^*$ is

$$\begin{aligned}\mathbf{E}^* &= \mathrm{grad}(\Delta\mathbf{x}^*) = (1 + \Delta s)\ \mathrm{grad}(\Delta\mathbf{x}) \\ &= (1 + \Delta s)\mathbf{E} = \mathbf{E} - \Delta s\ \mathbf{E}\end{aligned} \tag{19}$$

Note that $\Delta s\ \Delta\mathbf{x}$, and thus $\Delta s\ \mathbf{E}$, is only a second-order effect.

The worst-case effect can be estimated by considering a solution with very large strains of about $e = 1 \times 10^{-4}$ (100 ppm). If the datum is changed in scale by an extremely large amount, say $\Delta s = 1 \times 10^{-2}$ (10 000 ppm), the change in the deformation matrix is only $\Delta s \cdot e = 1 \times 10^{-6}$ (1 ppm). The strain elements are unlikely to exceed 50 ppm in a real geodetic network, in which case a scale change of over 20 000 ppm will be required to produce a 1-ppm change in strain. In practice, the scale of the network datum will generally be known to much better than 10-ppm accuracy, which would result in scale effects of only 0.001 ppm for this example. These estimates have been verified using numerical tests.

Concerning the invariance of strain under an orientation constraint, the problem is a bit more complicated. In fact, the symmetrical part of the strain matrix changes under the rotation of the coordinate system used for the adjustment, but the anti-symmetrical part does not. Neither do the strain parameters, called deformation measures in the following sections. We will show the proof of this after we have introduced the deformation measures.

4 Evaluation of the deformation matrix

The deformation matrix [Eq. (7)] for individual network points can be evaluated in a number of ways. The most simple approach is to obtain the partial derivatives directly from the displacements $\Delta\mathbf{x}$, obtained from Eq. (5). Let us take, for example, point $P_i = P_0$ with position vector $\mathbf{r}_i = (x_i, y_i) = \mathbf{r}_0$ and adjacent points P_j with position vectors $\mathbf{r}_j, j = 1, \ldots, 4$. For the point P_i and each point P_j, we can write two equations for two planes fitting the displacement components u_j and v_j as follows:

$$\forall j = 0, 1, \ldots, 4: \quad a_i + \left(\frac{\partial u_i}{\partial x}\right)(x_j - x_i) + \left(\frac{\partial u_i}{\partial y}\right)(y_j - y_i) = u_j \tag{20a}$$

$$\forall j = 0, 1, \ldots, 4: \quad b_i + \left(\frac{\partial v_i}{\partial x}\right)(x_j - x_i) + \left(\frac{\partial v_i}{\partial y}\right)(y_j - y_i) = v_j \tag{20b}$$

where all the partial derivatives as well as the absolute terms a_i, b_i and the coordinates x_i, y_i refer to point P_i. In matrix form, Eqs. (20) read

$$\forall j = 0, 1, \ldots, 4: \mathbf{K}_i \begin{bmatrix} a_i \\ \frac{\partial u_i}{\partial x} \\ \frac{\partial u_i}{\partial y} \end{bmatrix} = \mathbf{u}_i \tag{21a}$$

$$\forall j = 0, 1, \ldots, 4: \mathbf{K}_i \begin{bmatrix} b_i \\ \frac{\partial v_i}{\partial x} \\ \frac{\partial v_i}{\partial y} \end{bmatrix} = \mathbf{v}_i \tag{21b}$$

where $\mathbf{u}_i$ and $\mathbf{v}_i$ are sub-vectors of the total vector of displacements $\Delta\mathbf{x}$, which contain only the components referring to the five points in question. For $j \geq 1$ and points not on a line, Eq. (21) can be solved by using the method of least squares (LS). In solving for the unknown partial derivatives and absolute terms, all "observations" $\mathbf{u}$ and $\mathbf{v}$ are considered with equal weights. We then obtain

$$\forall i \text{ in the network: } \begin{bmatrix} a_i \\ \frac{\partial u_i}{\partial x} \\ \frac{\partial u_i}{\partial y} \end{bmatrix} = (\mathbf{K}_i^T\mathbf{K}_i)^{-1}\mathbf{K}_i^T\mathbf{u}_i = \mathbf{Q}_i\mathbf{u}_i \tag{22a}$$

$$\forall i \text{ in the network: } \begin{bmatrix} b_i \\ \frac{\partial v_i}{\partial x} \\ \frac{\partial v_i}{\partial y} \end{bmatrix} = (\mathbf{K}_i^T\mathbf{K}_i)^{-1}\mathbf{K}_i^T\mathbf{u}_i = \mathbf{Q}_i\mathbf{v}_i \tag{22b}$$

or

$$\forall i \text{ in the network: } \begin{bmatrix} a_i \\ \frac{\partial u_i}{\partial x} \\ \frac{\partial u_i}{\partial y} \\ b_i \\ \frac{\partial v_i}{\partial x} \\ \frac{\partial v_i}{\partial y} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_i \end{bmatrix} \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix} \tag{23}$$

Since the absolute terms are of no interest to us, we can eliminate the first row of the $\mathbf{Q}_i$ matrix and retain only the reduced matrix $\mathbf{Q}_i^*$ of dimensions $(n, 2)$, where $n - 1$ is the number of adjacent points used in evaluating the deformation matrix at point P_i. If we form appropriately a new matrix $\mathbf{T}_i$ containing the reduced matrices for the n points, we can use the full vector $\Delta\mathbf{x}$ on the right-hand side to obtain

$$\forall i \text{ in the network: } \begin{bmatrix} \frac{\partial u_i}{\partial x} \\ \frac{\partial u_i}{\partial y} \\ \frac{\partial v_i}{\partial x} \\ \frac{\partial v_i}{\partial y} \end{bmatrix} = \mathrm{vec}(\mathbf{E}_i) = \mathbf{T}_i\, \Delta\mathbf{x} \qquad (24)$$

Substituting $\Delta\hat{\mathbf{x}}$ from Eq. (5) for $\Delta\mathbf{x}$, we finally obtain

$$\forall i \text{ in the network: } \mathrm{vec}(\mathbf{E}_i) = \mathbf{T}_i(\mathbf{A}^T\mathbf{C}_l^{-1}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{C}_l^{-1}\Delta\mathbf{l} = \mathbf{L}_i\,\Delta\mathbf{l} \qquad (25)$$

which gives the matrix of deformation at P_i as a linear function of all the changes $\Delta\mathbf{l}$ in the observables.

For readers familiar with numerical methods, we note that the described technique is nothing other than a particular application of finite difference method for the evaluation of the partial derivatives of interest. The selection of the "neighboring" points for the evaluation of the deformation matrix at P_i may be done by following different principles. The simplest, and most geometrically meaningful, is to use either all or the closest of the points connected by observations to be the points of interest P_i (Vaníček et al. 1991).

It should be pointed out that we may, and do, encounter singular cases of Eq. (19). Apart form the unlikely case, when the matrix of normal equations $\mathbf{A}^T\mathbf{C}_l^{-1}\mathbf{A}$ is singular and there are infinitely many solutions $\mathrm{vec}(\mathbf{E}_i)$, we obtain situations where the strain matrix at a specific point (or a set of points) is undefined. This happens when the points from which the strain components are evaluated [cf. Eqs. (20)] are all in one line. It is not possible to fit a plane to such a set of points and the strain components in the direction of coordinate axes become undefined. In fact, it is possible to compute just one strain component, in the direction of the line formed by the defining points, but we consider the discussion of the treatment of such singularity to be beyond the scope of this paper.

Another pathological case should be discussed here, however, as it does occur in practice. When a point is tied to the other points of the network by only one observation line, we can form only two observation equations (one for the x component and one for the y component) and the strain matrix at this point is again undetermined. All the notions mentioned in the above paragraph then apply to this case as well. The points connected by only one observation link are therefore omitted from the robustness analysis.

5 Deformation measures

According to Eq. (25), every potential change Δl_i in an observation causes a potential deformation of the whole network. This means that all the deformation matrices at all the network points (each caused by a change of one observation) will be generally different from zero. For m observations within the network we will have m different deformation matrices for each point. In order to study the degree of deformation caused by potential gross errors in observations, it is necessary to consider only the largest deformation at each point. This largest potential deformation corresponds to the weakest link in the network – the network can only be as strong or robust as its weakest link. This is our "credo of the weakest link" (Dare 1983) which we use systematically in measuring potential network deformation.

Since the deformation at each point is described by a matrix, the distinction of "the largest" deformation is not a trivial problem. It would be necessary to associate a scalar measure with each matrix and use that measure to recognize the largest deformation. This, however, would not make much sense from a geometrical point of view because the deformation matrix describes at least three measures of deformation: strain, shear, and the already discussed differential rotation. These measures are more or less independent and as such they must be considered separately.

In our approach (Vaníček et al. 1991) we use the following descriptors (deformation measures): mean strain or dilation σ, total shear γ, and the above introduced local differential rotation ω. Mean strain is equal to one-half of the deformation matrix trace

$$\sigma = \tfrac{1}{2}\mathrm{tr}(\mathbf{E}) = \tfrac{1}{2}\mathrm{tr}(\mathbf{S}) = \tfrac{1}{2}(\partial u/\partial x + \partial v/\partial y) \qquad (26)$$

Complete shear is defined as the geometric mean of pure shear τ

$$\tau = \tfrac{1}{2}(\partial u/\partial x - \partial v/\partial y) \qquad (27)$$

and simple shear υ

$$\upsilon = \tfrac{1}{2}(\partial u/\partial y + \partial v/\partial x) \qquad (28)$$

so that we have

$$\gamma = \tfrac{1}{2}\sqrt{\tau^2 + \upsilon^2} \qquad (29)$$

As we have stated above, these three deformation measures are invariant under an arbitrary rotation of the coordinate system, and thus under an arbitrary choice of the orientation constraint or orientation datum. The proof of the invariance is a bit involved and is shown in the Appendix in order not to interrupt the flow of the argument here.

Deformation measure can, of course, be quantified by many other means. Our description has the advantage of having an easy intuitive interpretation; mean strain can be regarded as a deformation in scale, complete shear as deformation of (local) configuration, and local differential rotation (corrected for the mean) as local twisting.

6 Robustness

As soon as we decide how to "measure" potential network deformation, it becomes easy to measure the degree of network robustness. All we have to do is substitute maximum undetectable observational errors from Eq. (3) for the arbitrary changes in observations in Eq. (25) to obtain

$$\text{vec}(\mathbf{E}_i) = \lambda(\alpha_o, \beta_o)\mathbf{L}_i \frac{\sigma_{l_i}}{\sqrt{r_i}} \tag{30}$$

where α_o is the significance level used for testing the outliers in the network, and then follow the algorithm described in the previous paragraph. At each point the three values of deformation parameters (primitives or measures) are computed from the maximum undetectable errors in all observations; i.e. $3n$ values per point for all n points. If desired, the observations responsible for the largest values of deformation measures can also be easily identified.

The maximum absolute value of each deformation primitive at the point is the measure of robustness of the network, i.e. network robustness is characterized by three real numbers for each point. These numbers can be plotted in three maps to give us a visual aid. We speak of network robustness in scale, robustness in configuration and robustness in twist. It makes sense to plot the parameters in parts per million. These units make it easy to compare robustness measures against relative errors in observations which are normally also expressed in parts per million. We note that the larger the absolute value of the deformation parameter, the less robust is the network at that point; a robust network will display small values of deformation primitives.

It should be obvious from Eq. (30) that the location of the extreme robustness points (points of maximum and minimum robustness) is independent of the choice of α_o or β_o. The non-centrality parameter λ_o, whose magnitude depends on this choice, is a multiplier common to all three deformation primitives and thus controls only the magnitudes of those primitives without affecting their relative behavior. Realistic choice of these two probabilities will become important when we want to design tolerance limits for network robustness, or rather the lack of it.

7 Examples

In order to illustrate the differences between traditional covariance analysis and robustness analysis, we have examined a simple synthetic horizontal network, shown in Fig. 2. The network consists of 11 points, one of which (point 1) is fixed, 19 distances, 38 directions, and 1 azimuth. The distances were assigned realistic standard deviation of 3 mm + 2 ppm while the directions were assigned a standard deviation of 0.5″. The datum orientation was defined by the azimuth with a standard deviation of 1″.

The traditional covariance analysis of the propagation of random errors is depicted by the absolute and relative confidence ellipses shown in Figs. 2 and 3, respectively. These ellipses are based on a significance level of 5% (confidence level of 95%). The absolute error ellipses show the propagation of errors outward from the fixed point (1) in the network. From the orientation of the ellipses, the distances are slightly more accurate than the directions – the ellipses are smaller along the direction to the fixed point. The relative ellipses show similar characteristics locally. The error ellipses are all

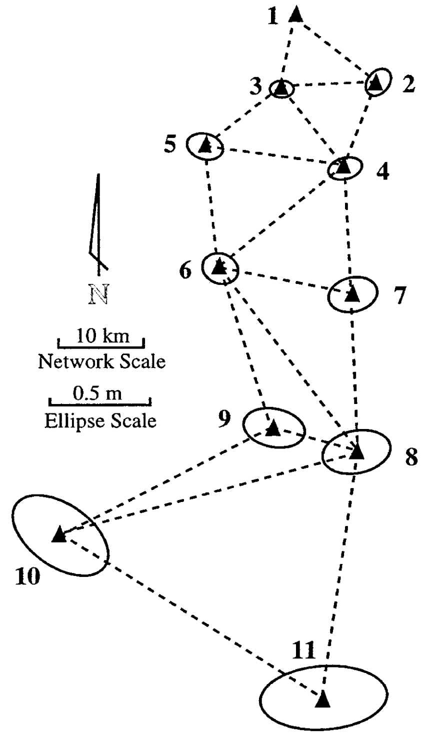


Fig. 2. Absolute 95% confidence ellipses. Dashed lines represent observed lines (distances and/or directions) in the network

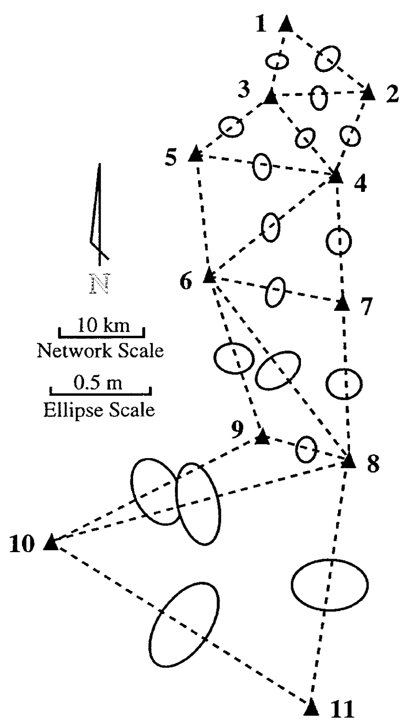


Fig. 3. Relative 95represent observed lines (distances and/or directions) in the network

flattened along the line connecting the end points, indicating that the distances are more precise than the directions. Again, the ellipses are larger for longer distances, as expected. From this covariance analysis it appears that only the bottom of the network is significantly weaker in terms of random errors than the other parts, primarily due to the longer distances between points. We have included the results of covariance analysis here to illustrate numerically the fact that it shows a different aspect of the network from that of robustness. As covariance and robustness analyses address different aspects of the network, nothing more should be read into this comparison.

The robustness analysis quantifies the propagation of potentially undetected blunders. This is depicted by the three deformation primitives in Figs. 4, 5, and 6. All three figures indicate that the top and bottom areas of the network are the weakest in terms of the effect of potential blunders on the coordinates estimates. Errors of approximately 5–6 ppm in orientation, configuration, and scale can be expected in these areas. Notice that the smallest robustness is obtained for point 1, the fixed point. The center part of the network is clearly the strongest, primarily due to the larger number and more favorable geometry of observations among these points. Clearly the greater observational redundancy in this area makes these points more resistant to potential blunders.

In order to illustrate how robustness analysis works with real networks, we show another example from the Canadian province of Newfoundland. This time we have chosen a real global positioning system (GPS) network consisting of 104 points with 786 observed coordinate differences – see Fig. 7. The robustness measures are shown in Figs. 8–10 and summarized in Table 1.

First, we may observe that the network as a whole is quite robust, as we might expect. Next, we note that robustness of points 11, 69, and 70 is undefined. This is because these points are linked to the rest of the points by only one observation tie – see the discussion at the end of Sect. 4. Finally, we note that the southwestern part of the network is the least robust in all three robustness measures. It is only by coincidence that one and the same point (78) is the least robust in all three measures. Most often, the points least robust in the three measures are different.

8 Conclusions and recommendations

The combination of reliability analysis with geometric strength analysis has resulted in the conception of a new technique, robustness analysis, which is a natural merger of the two existing techniques. Experiences with robustness analysis show that it is a very powerful technique, capable of providing a picture of the analyzed network which is complementary to the one furnished by the standard covariance analysis. "Network robustness"

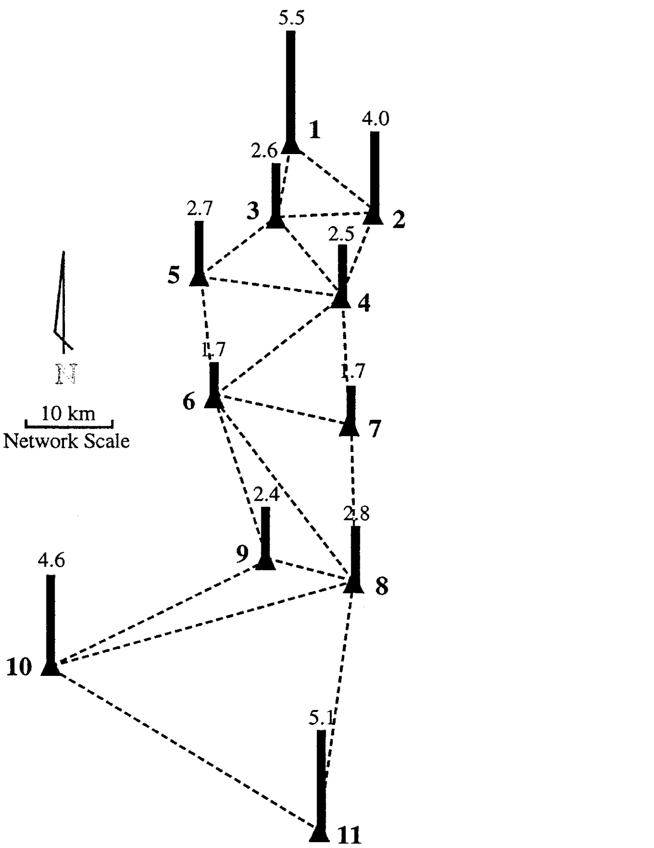


Fig. 4. Robustness in orientation (differential rotation) in ppm. Dashed lines represent observed lines (distances and/or directions) in the network

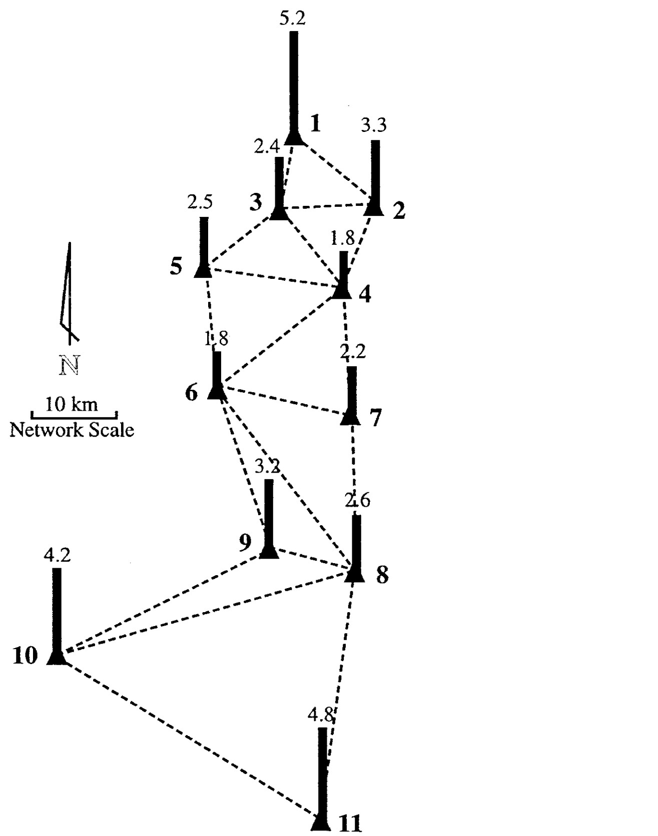


Fig. 5. Robustness in configuration/shape (shear) in ppm. Dashed lines represent observed lines (distances and/or directions) in the network

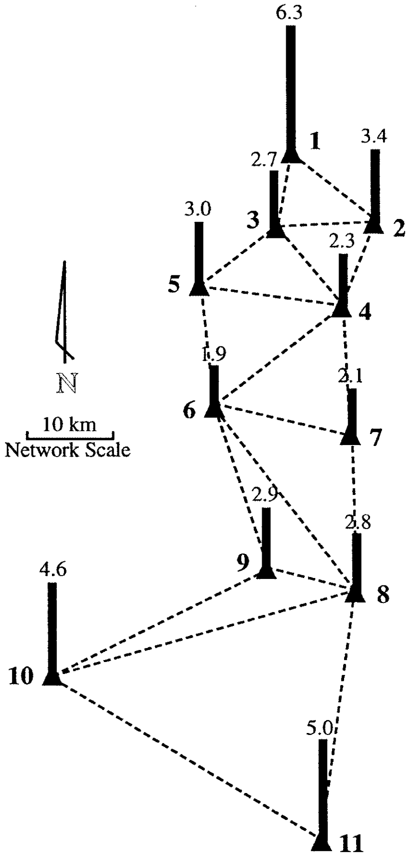
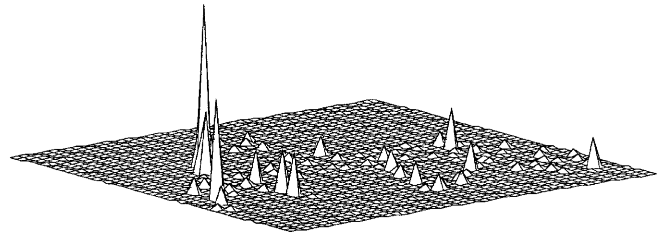
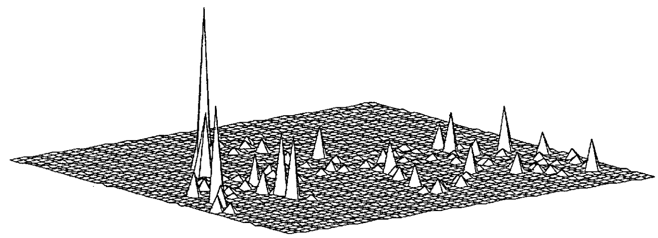


Fig. 6. Robustness in scale (dilation) in ppm. Dashed lines represent observed lines (distances and/or directions) in the network



Max Twist Strain = 25.69 ppm, Point # 78

Fig. 8. Surface plot of robustness in twist for Newfoundland GPS network



Max Shear Strain = 27.77 ppm, Point # 78

Fig. 9. Surface plot of robustness in shear for Newfoundland GPS network

(strength, as an ability to resist deformations induced by undetectable blunders, might be a term more readily understood) is invariant with respect to coordinate shifts and orientation, and almost invariant with respect to scale changes.

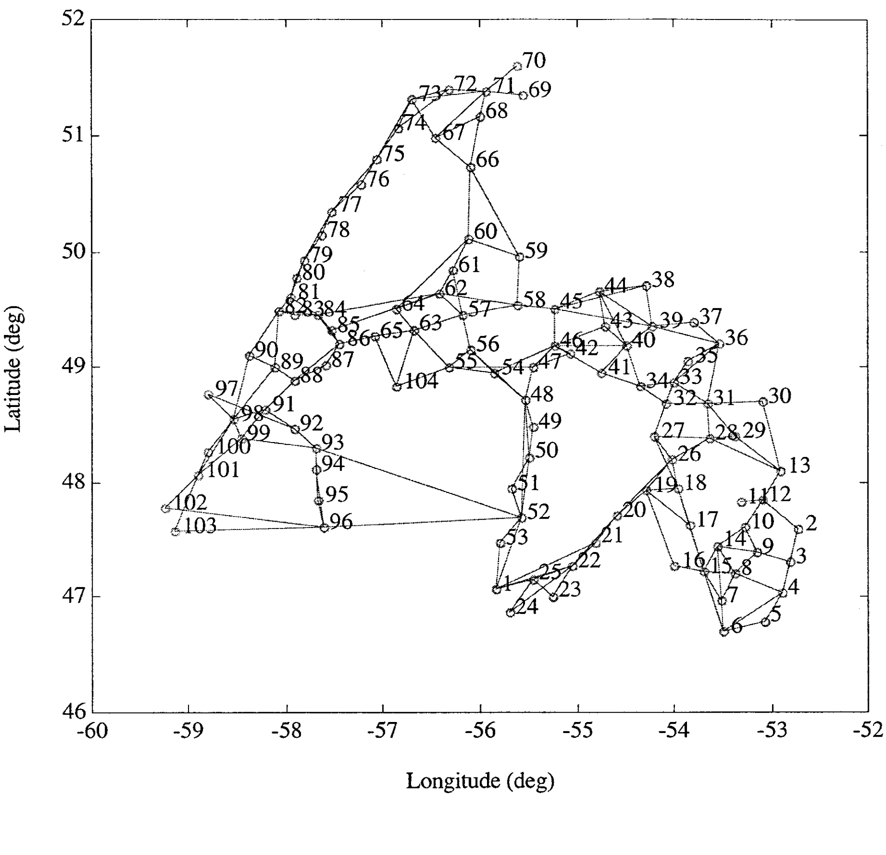
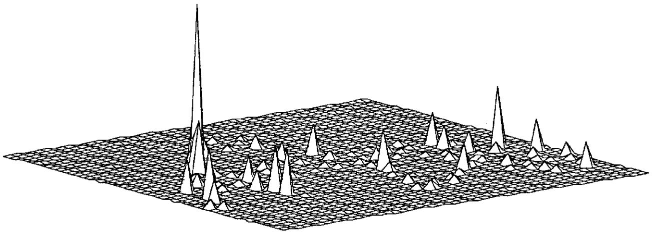


Fig. 7. Newfoundland GPS network. Connections between points represent observed GPS baselines



Max Scale Strain = 38.51 ppm, Point # 78

Fig. 10. Surface plot of robustness in scale for Newfoundland GPS network

Robustness is expressed in terms of three independent deformation measures, namely, robustness in scale (dilation), local configuration (shear), and twist (differential rotation). It thus makes no sense to talk about robustness in general but only about "robustness in scale", "robustness in shear", and "robustness in twist". This will sound complicated to a surveyor uninitiated in the concepts of deformation analysis, where the three primitives are used routinely. Let us emphasize here that the full description of a deformation cannot be achieved with fewer than three measures. If we wish to deal with network strength meaningfully, then we have to accept this fact and learn to live with it. It seems to us that the introduction of robustness analysis will require some educational effort; specifically, a guide/manual will have to be written with the aim to assist in the transfer of knowledge. We recommend that robustness analysis be used side-by-side with the standard covariance analysis for a complete network analysis in the future, and that national specifications for accuracy standards be extended to include robustness analysis.

As we have seen, it is not always easy, or even possible, to guess at the reason behind a specific weakness in the network from the network configuration alone. More experiments should be conducted with robustness analysis, and more experience gained with practical applications as well as the interpretation of robustness analysis results, particularly before specific values of robustness tolerance limits can be imposed through specifications. Some general criteria, however, can already be formulated, and these have been spelled out above. The robustness of a planned network, a robustness pre-analysis, may prove to be more important than a post-analysis of an already established network. A better graphical representation of robustness measures is a must.

A strategy is being worked out to deal with the two kinds of singularities that may arise in robustness analysis. While the generic singularity associated with the extreme weakness of the network has so far been shown by relatively large values of the robustness primitives, geometrical singularities have been simply eliminated by leaving out the singular points and noting them in the output. More worrisome is the case of geometrical near-singularities. A measure of ill-conditioning based either on confidence regions for strength measures or on the value of the determinant in the LS

Table 1. Robustness measures for Newfoundland GPS network

Point	Twist (ppm)	Shear (ppm)	Scale (ppm)
1	−0.8	0.8	1.0
2	1.0	1.6	2.9
3	−1.0	1.2	1.5
4	0.8	1.0	1.5
5	4.7	5.2	5.6
6	0.6	0.9	1.5
7	−0.7	1.1	1.4
8	0.6	0.7	0.9
9	0.5	0.5	0.9
10	0.7	0.8	1.1
11	undefined	undefined	undefined
12	1.0	1.1	2.1
13	−0.8	3.8	7.2
14	−0.5	0.6	1.0
15	−0.7	1.2	1.0
16	−1.1	3.3	1.0
17	−0.7	1.4	2.7
18	−0.6	0.9	1.2
19	0.6	0.6	1.0
20	4.5	4.9	5.9
21	−1.0	1.0	1.3
22	1.0	1.0	1.1
23	1.9	1.9	1.6
24	−2.3	2.3	2.2
25	−1.1	1.2	1.2
26	−0.8	0.8	0.9
27	−0.4	0.9	1.8
28	0.4	0.5	0.8
29	0.5	1.2	2.2
30	1.1	7.3	14.2
31	1.1	2.6	4.8
32	0.5	0.8	1.1
33	0.7	0.7	0.9
34	−0.4	0.8	1.0
35	−6.3	6.5	5.4
36	−0.5	0.6	1.0
37	−2.3	4.2	7.6
38	0.5	0.7	1.3
39	−0.3	0.4	0.7
40	0.4	0.4	0.6
41	0.7	0.8	0.9
42	1.9	2.1	2.5
43	−0.6	0.7	1.1
44	0.3	0.4	0.8
45	0.4	0.4	0.6
46	−0.4	0.4	0.6
47	−0.8	0.9	1.1
48	0.4	0.7	0.9
49	3.2	5.4	9.7
50	−1.0	1.4	1.9
51	−1.7	2.1	1.0
52	−0.2	0.5	0.4
53	−3.6	3.9	2.3
54	0.5	0.8	0.9
55	0.4	0.8	0.4
56	0.4	1.5	1.2
57	0.6	0.7	0.9
58	−1.0	1.2	1.8
59	−3.0	5.2	6.9
60	1.0	1.0	1.2
61	1.5	2.2	4.4
62	0.5	0.6	0.9
63	0.6	0.7	0.6
64	−1.2	1.3	2.2
65	0.8	1.0	1.1
66	−1.1	2.1	3.1
67	0.9	1.0	1.0

Table 1. (Contd)

Point	Twist (ppm)	Shear (ppm)	Scale (ppm)
68	−1.1	1.1	0.8
69	undefined	undefined	undefined
70	undefined	undefined	undefined
71	1.5	1.5	1.3
72	−0.9	1.1	1.5
73	1.6	1.7	2.0
74	1.4	1.5	2.4
75	−3.4	3.7	4.1
76	12.1	12.3	14.3
77	−4.9	5.2	4.1
78	**25.7**	**27.8**	**38.5**
79	5.5	6.0	8.6
80	−9.9	11.2	12.4
81	0.5	0.6	0.9
82	0.6	0.7	1.2
83	−0.7	1.2	2.3
84	−0.5	0.6	0.7
85	−0.8	0.9	1.2
86	0.6	0.8	1.4
87	5.0	5.1	6.0
88	−1.3	1.3	1.2
89	−0.5	0.6	1.0
90	−1.5	1.9	3.0
91	−1.1	1.1	1.1
92	0.9	3.7	4.6
93	−0.5	1.2	1.1
94	−6.7	10.7	10.6
95	−6.7	10.6	10.6
96	−0.5	0.8	0.8
97	−2.5	4.3	7.7
98	−1.2	1.4	1.1
99	−0.7	0.7	0.8
100	−15.9	16.4	12.3
101	2.9	3.6	5.3
102	−0.9	2.8	1.9
103	−0.7	1.5	1.9
104	0.8	0.9	1.0

fitting of planes in the determination of strain matrices should be devised.

Some refinement of the reliability analysis as the first part of robustness analysis is also called for in order to understand better the role of the probabilities (significance levels) used in the univariate and multivariate tests and their impact on the non-centrality parameter λ. An attempt to understand the total picture of how those probabilities work was made and illustrated with numerical examples in Craymer et al. (1993a,b, 1995), Szabo et al. (1993), and Krakiwsky et al. (1993, 1999). Even though a thoughtful selection of the β_0 probability was not necessary in our investigations – β_0 affects only the scale of the robustness plots – it will become necessary for formulating the robustness tolerance limits. This point should also be further investigated.

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Appendix

Proof of invariance of strain primitives under rotation of coordinate system.

Consider two analyses of a network, the first using coordinate system (x, y) where [cf. Eqs. (6) and (7)]

$$\Delta\mathbf{x}_i = \begin{bmatrix} \Delta x_i \\ \Delta y_i \end{bmatrix} = \begin{bmatrix} u_i \\ v_i \end{bmatrix} \tag{A1}$$

$$\mathbf{E}_i = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix}_i \tag{A2}$$

The second analysis uses coordinate system (x^*, y^*), rotated by Ω with respect to (x, y), so that

$$\Delta\mathbf{x}_i^* = \begin{bmatrix} u_i^* \\ v_i^* \end{bmatrix} = \mathbf{R}(\Omega)\begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} \cos\Omega\, u_i + \sin\Omega\, v_i \\ -\sin\Omega\, u_i + \cos\Omega\, v_i \end{bmatrix} \tag{A3}$$

$$\mathbf{E}_i^* = \begin{bmatrix} \partial u^*/\partial x^* & \partial u^*/\partial y^* \\ \partial v^*/\partial x^* & \partial v^*/\partial y^* \end{bmatrix}_i \tag{A4}$$

Now, using the chain rule for differentiation

$$\partial u^*/\partial x^* = \partial u^*/\partial x \cdot \mathrm{d}x/\mathrm{d}x^* + \partial u^*/\partial y \cdot \mathrm{d}y/\mathrm{d}x^* \tag{A5}$$

where

$$\mathrm{d}x/\mathrm{d}x^* = \cos\Omega, \quad \mathrm{d}y/\mathrm{d}x^* = \sin\Omega \tag{A6}$$

and

$$\partial u^*/\partial x = \cos\Omega\,\partial u/\partial x + \sin\Omega\,\partial v/\partial x \tag{A7}$$

$$\partial u^*/\partial y = \cos\Omega\,\partial u/\partial y + \sin\Omega\,\partial v/\partial y \tag{A8}$$

Therefore

$$\begin{aligned}\partial u^*/\partial x^* = {} & \cos^2\Omega\,\partial u/\partial x - \cos\Omega\,\sin\Omega\,\partial v/\partial x \\ & + \cos\Omega\,\sin\Omega\,\partial u/\partial y + \sin^2\Omega\,\partial v/\partial y\end{aligned} \tag{A9}$$

Similarly, for the other differentials

$$\begin{aligned}\partial u^*/\partial y^* = {} & -\cos\Omega\,\sin\Omega\,\partial u/\partial x - \sin^2\Omega\,\partial v/\partial x \\ & + \cos^2\Omega\,\partial u/\partial y + \cos\Omega\,\sin\Omega\,\partial v/\partial y\end{aligned} \tag{A10}$$

$$\begin{aligned}\partial v^*/\partial x^* = {} & -\cos\Omega\,\sin\Omega\,\partial u/\partial x + \cos^2\Omega\,\partial v/\partial x \\ & - \sin^2\Omega\,\partial u/\partial y + \cos\Omega\,\sin\Omega\,\partial v/\partial y\end{aligned} \tag{A11}$$

$$\begin{aligned}\partial v^*/\partial y^* = {} & \sin^2\Omega\,\partial u/\partial x - \cos\Omega\,\sin\Omega\,\partial v/\partial x \\ & - \cos\Omega\,\sin\Omega\,\partial u/\partial y + \cos^2\Omega\,\partial v/\partial y\end{aligned} \tag{A12}$$

Substituting these expressions into the anti-symmetric part of the strain matrix in the rotated coordinate system gives [cf. Eq. (11)]

$$A^* = \begin{bmatrix} 0 & -\omega^* \\ \omega^* & 0 \end{bmatrix} \tag{A13}$$

where

$$\omega^* = \frac{1}{2}(\partial v^*/\partial x^* - \partial u^*/\partial y^*) = \frac{1}{2}(\partial v/\partial x - \partial u/\partial y) = \omega \tag{A14}$$

The anti-symmetric strain matrix and thus even the differential rotation are therefore invariant under a rotation of the coordinate system.

The mean shear (σ) and complete shear (γ) strain primitives can also be shown to be invariant under rotation. Mean strain transforms as

$$\sigma^* = \frac{1}{2}(\partial u^*/\partial x^* + \partial v^*/\partial y^*) = \frac{1}{2}(\partial u/\partial x + \partial v/\partial y) = \sigma \tag{A15}$$

Pure and simple shear read, in the rotated coordinate system

$$\tau^* = \frac{1}{2}(\partial u^*/\partial x^* - \partial v^*/\partial y^*) = \cos^2\Omega\,\tau + \sin^2\Omega\,\nu \tag{A16}$$

$$\nu^* = \frac{1}{2}(\partial u^*/\partial y^* + \partial v^*/\partial x^*) = \cos^2\Omega\,\nu - \sin^2\Omega\,\tau \tag{A17}$$

Complete shear then transforms as

$$\gamma^* = \frac{1}{2}\sqrt{\tau^{*2} + \nu^{*2}} = \frac{1}{2}\sqrt{\tau^2 + \nu^2} = \gamma \tag{A18}$$

Thus, all three strain primitives (ω, σ, γ) are invariant under a rotation of the coordinate system.

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